# Complex Numbers

## The number "i"

$i$ is defined by:

$i^2 = -1$ or $i = \sqrt{-1}$

This is called the **imaginary number**. Sometimes electrical engineers use $j$ to mean the same thing because $i$ is traditionally electric current for them.

## Complex numbers

**Complex numbers** are expressions comprising a real number added to some multiple of $i$. For example:

$2+5i$ or $3-7i$

Or if you want to be formal: $a+bi$

So a complex number can be said to have a **real part** ($a$) and an **imaginary part** ($bi$).

## Arithmetic

Ordinary maths with complex numbers is mostly easy – you just add, multiply and subtract as you would any other expressions. The only important thing to remember is that $i^2 = -1$ which you can simplify wherever you see it.

**Division** is slightly harder because if you end up with a complex number as your divisor you appear to be stuck for simplifications – e.g.

$$\frac{3}{4i}$$

But in this case you can apply the good old "multiply the top and bottom by the same number" technique (using $i$ in this case) to return things to sanity:

$$\frac{3}{4i} \times \frac{i}{i} = \frac{3i}{4i^2} = -\frac{3i}{4}$$

Another more difficult case is where you have something like:

$$\frac{3}{4+2i}$$

i.e. more than one term in the divisor. You can't just multiply everything by $i$ because you'd end up with $4i$ in the divisor which solves nothing.

What you can do, however, is take advantage of the difference of two squares property by multiplying everything by something called the **complex conjugate**.

### Complex Conjugate

The complex conjugate of a complex number $a+bi$ is just $a-bi$ or in other words to get the conjugate you just change the sign.

Why's this useful? Well because it allows us to get rid of $i$ if we multiply them by each other. Watch this...

$(a+bi)(a-bi) = a^2 + abi - abi - (bi)^2$

$= a^2 - (bi)^2$

$= a^2 - b^2 \times -1$

$= a^2 + b^2$

$i$ has disappeared!

So back to our example, we can use the complex conjugate to make the divisor real:

$$\frac{3}{4+2i} = \frac{3(4-2i)}{(4+2i)(4-2i)} = \frac{3(4-2i)}{4^2+2^2} = \frac{12-6i}{20} = \frac{6-3i}{10}$$

# Geometry

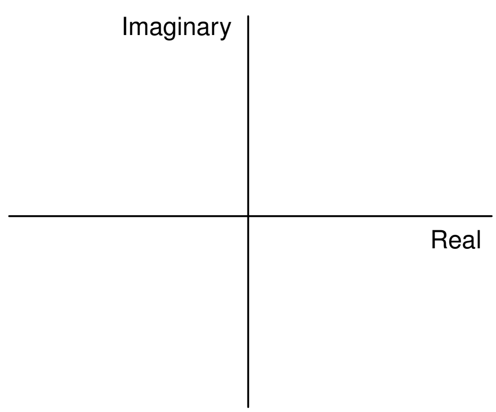
Complex numbers have some geometric interpretations that make them useful for solving other sorts of problems, particularly relating to sound and DSP.

## Complex Plane

The **complex plane** is like the usual 2D $xy$ plane you're used to but instead of $x$ and $y$ axes we have the **real** and **imaginary** axes. You can then plot complex numbers geometrically quite easily – by plotting the two parts of the number against their respective axes.

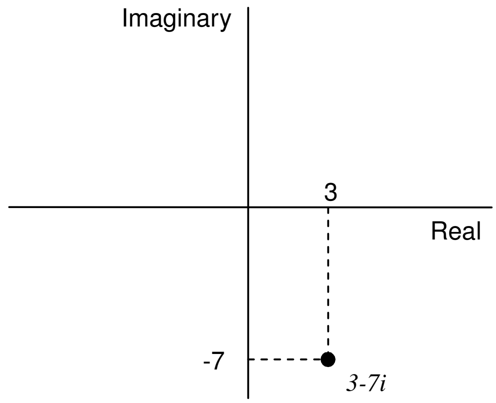
Suppose we want to plot $3-7i$

Here's our empty complex plane:




(by convention the Real axis is the horizontal one)

We plot $3-7i$ like this:




Yes it is that easy; the *a* portion is the real value the *b* portion is the imaginary value.

### Magnitude

You can compute the magnitude of a complex number – imagine this as the length of the line from the origin to the point on our plot above. To do this use the distance formula (derived from Pythagoras):

$$|3-7i| = \sqrt{3^2+7^2} = \sqrt{58} \approx 7.6$$

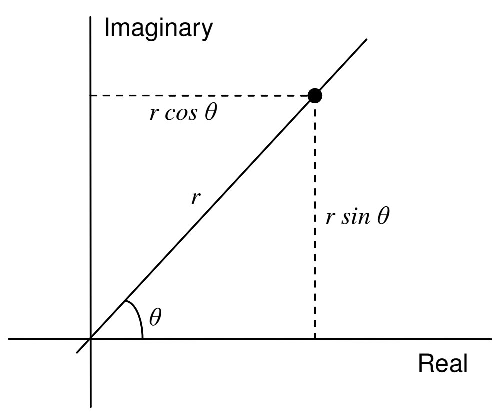
Note that *i* does not feature in our actual calculation – this is because we're assumed to be working entirely in the complex plane now; the complex magnitude has no meaning in our normal *xy* plane so we don't need to qualify it with *i*.

This is about the only way you can **compare** complex numbers, by the way – just as it's the only way to compare Cartesian coordinates.

### Polar Coordinates

Unsurprisingly, if we can plot a point on the complex plane using some coordinates we ought to be able to specify the position in **polar** coordinates too. That is, using coordinates comprising an angle $\theta$ from the real axis (also called the **argument**) and a **magnitude** $r$.

We just saw how the magnitude works, given the complex number. What about the argument? To calculate this you just need some trigonometry.




So look: going back to our earlier notation of $a+bi$ to represent a complex number, it is clear that this could now be written (by letting $a = r\cos\theta$ and $b = r\sin\theta$) as:

$$r\cos\theta + ir\sin\theta$$

This is the **polar form of a complex number**.

## Euler's Formula

Leonhard Euler, a Swiss guy, came up with the following formula:

$$\cos\theta + i\sin\theta = e^{i\theta}$$

This can be proved by differentiation, but let's not bother and assume it's true. This now allows us to write our complex numbers in terms of $e$ like so:

$$r\cos\theta + ir\sin\theta = re^{i\theta}$$

Also note that the complex conjugate $z^*$ can be written as:

$$z^* = re^{-i\theta}$$

...since all that's really happening to go from $a+bi$ to $a-bi$ is we're negating $i$, which can carry through to the above.

Note that geometrically the complex conjugate is just a **reflection** in the real axis, with the same magnitude.